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A re-examination of the quantum theory of optical cavities with moving mirrors

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Abstract. The quantization of linearly polarized light in an ideal optical cavity with moving boundaries is re-examined. The mathematical results of the earlier study by Moore are obtained using more conventional techniques. In contrast to Moore, it is shown that the theory does possess a Hamiltonian and a Schrödinger picture when any suitable Lagrangian coordinate system is chosen. It is demonstrated that the quantum, as well as the classical, physics is independent of the choice of Lagrangian coordinate system.

1. Introduction

The quantization of the electromagnetic field in an ideal optical cavity with moving boundaries was first discussed by Moore [1] in 1970, and his treatment forms the basis for current work. The primary field of interest is now Unruh radiation, in this context the creation of photons from the vacuum by the movement of the cavity mirrors [2–4]. Other fields involving similar systems include the Casimir and vacuum radiation pressure effects [5–7], interferometry in gravity wave detection [8], and laser physics [1, 9].

Moore's quantization method involves introducing a symplectic structure on the space of classical solutions [1, 2] and it is not immediately clear how this relates to more conventional techniques. Moore also claims that the system has no Hamiltonian and no Schrödinger picture, which upon closer examination is not entirely correct, as we will demonstrate. In this paper we will quantize the moving-mirror cavity using the usual Lagrangian procedure in a suitable coordinate system and show that the quantization does not essentially depend on the choice of coordinate system. We will then independently construct the p_n and q_n quantum operators used in Moore's work and show that they have the properties required of them.

2. Classical theory

We will consider an electric field polarized in the z direction in a one-dimensional cavity with perfect mirrors at $x = 0$ and $x = q(t)$. The classical wave equation is [1, 2]

$$\frac{\partial^2 A}{\partial t^2} = \frac{\partial^2 A}{\partial x^2} \tag{1}$$

with boundary conditions

$$A(0) = A(q(t)) = 0. \tag{2}$$

We will generalize this slightly to include the case, examined by Law [4], in which there is a time- and space-dependent dielectric within the moving-mirror cavity. The wave equation is now

$$\frac{\partial}{\partial t}\left[\varepsilon(x,t)\frac{\partial A}{\partial t}\right]=\frac{\partial^2 A}{\partial x^2} \tag{3}$$

and the Lagrangian is

$$\mathcal{L}=\frac{1}{2}\int_0^{q(t)}\mathrm{d}x\left[\varepsilon(x,t)\left(\frac{\partial A}{\partial t}\right)^2-\left(\frac{\partial A}{\partial x}\right)^2\right]. \tag{4}$$

3. Quantum theory

The coordinate system $A(x,t)$ is unsuitable for the quantization of the electromagnetic field, because the $A(x,t)$ do not represent degrees of freedom for all times t. It is essential therefore to transform to a suitable coordinate system $B(y,t)$ where the boundary conditions are fixed, i.e.

$$B(0,t)=B(L,t)=0. \tag{5}$$

As we will demonstrate, in this new coordinate system there is a Hamiltonian and an associated Schrödinger picture. Although these depend on the choice of coordinates, the Heisenberg picture, and hence the actual physics of the system, do not.

Note that we are not considering a general or special relativistic coordinate transformation, only a transformation of the Lagrangian coordinates. Although the time t is unchanged by the transformation it is more general than that considered previously [10]. Other work concerning Unruh radiation [10, 11] suggests that quantum physics may be changed by coordinate transformations which affect both x and t.

We take a well-behaved coordinate transformation

$$B(y,t)=A(x(y,t),t) \tag{6}$$

$$\varepsilon_B(y,t)=\varepsilon(x(y,t),t) \tag{7}$$

where

$$x(0,t)=0 \tag{8}$$

$$x(L,t)=q(t) \tag{9}$$

for all t.

Formally, we will require $x(y,t)$ to be strictly increasing in y for constant t and continuous in both y and t. The first derivatives of x must exist at least piecewise and be bounded (though not necessarily be continuous). It is convenient for simplicity to assume that x is differentiable everywhere to second order, although this is not strictly necessary if x is treated as a generalized function.

It is well known that the classical physics derived from the Lagrangian are unaffected by such a coordinate transformation. We will now show that this is also true of the quantum physics.

On using the chain rule we have

$$\frac{\partial B}{\partial y}=\frac{\partial A}{\partial x}\frac{\partial x}{\partial y} \tag{10}$$

and

$$\frac{\partial B}{\partial t}=\frac{\partial A}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial A}{\partial t}. \tag{11}$$

Similar relations for $\partial \varepsilon_B/\partial y$ and $\partial \varepsilon_B/\partial t$ allow us to write the system Lagrangian

$$\mathcal{L} = \frac{1}{2}\int_0^{q(t)} \mathrm{d}x \left[\varepsilon\left(\frac{\partial A}{\partial t}\right)^2 - \left(\frac{\partial A}{\partial x}\right)^2\right]$$

as

$$\frac{1}{2}\int_0^{L} \mathrm{d}y \left\{\varepsilon_B\left(\frac{\partial B}{\partial t}\right)^2 \frac{\partial x}{\partial y} - \varepsilon_B \left\{\frac{\partial B}{\partial t}, \frac{\partial B}{\partial y}\right\}\frac{\partial x}{\partial t} + \left(\frac{\partial B}{\partial y}\right)^2\left[\varepsilon_B\left(\frac{\partial x}{\partial t}\right)^2 - 1\right]\left(\frac{\partial x}{\partial y}\right)^{-1}\right\}$$

where $\{O_1, O_2\}$ is the anticommutator $O_1O_2 + O_2O_1$.

This results in a canonical momentum

$$\Pi_B = \frac{\delta\mathcal{L}}{\delta\dot{B}} = \varepsilon_B\frac{\partial B}{\partial t}\frac{\partial x}{\partial y} - \varepsilon_B\frac{\partial B}{\partial y}\frac{\partial x}{\partial t} = \varepsilon\frac{\partial A}{\partial t}\frac{\partial x}{\partial y} \tag{12}$$

and hence a canonical commutation relation

$$[B(y,t), \Pi_B(y_0,t)] = \mathrm{i}\delta(y-y_0) = \mathrm{i}\frac{\partial x}{\partial y}\delta(x(y,t) - x(y_0,t)) \tag{13}$$

or, in terms of A,

$$\left[A(x(y,t),t), \varepsilon\frac{\partial A}{\partial t}(x(y_0,t),t)\right] = \mathrm{i}\delta(x(y,t) - x(y_0,t)). \tag{14}$$

The Hamiltonian for this particular coordinate system is given by

$$\begin{aligned}\mathrm{H}_B &= \frac{1}{2}\int_0^L \left\{\Pi_B, \frac{\partial B}{\partial t}\right\}\mathrm{d}y - \mathcal{L}\\ &= \frac{1}{2}\int_0^L \mathrm{d}y \left\{\varepsilon_B\left(\frac{\partial B}{\partial t}\right)^2\frac{\partial x}{\partial y} - \left(\frac{\partial B}{\partial y}\right)^2\left[\varepsilon_B\left(\frac{\partial x}{\partial t}\right)^2 - 1\right]\left(\frac{\partial x}{\partial y}\right)^{-1}\right\}\end{aligned} \tag{15}$$

and then substituting

$$\frac{\partial B}{\partial t} = \varepsilon_B^{-1}\Pi_B\left(\frac{\partial x}{\partial y}\right)^{-1} + \frac{\partial B}{\partial y}\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial x}{\partial t} \tag{16}$$

gives

$$\mathrm{H}_B = \frac{1}{2}\int_0^L \mathrm{d}y \left(\frac{\partial x}{\partial y}\right)^{-1}\left\{\varepsilon_B^{-1}\Pi_B^2 + \left\{\Pi_B, \frac{\partial B}{\partial y}\right\}\frac{\partial x}{\partial t} + \left(\frac{\partial B}{\partial y}\right)^2\right\}.$$

As a consistency check, we will calculate the evolution of B:

$$\begin{aligned}\frac{\partial B}{\partial t}(y_0,t) &= \mathrm{i}[\mathrm{H}_B, B(y_0,t)]\\ &= \int_0^L \mathrm{d}y \left(\frac{\partial x}{\partial y}\right)^{-1}\left\{\varepsilon_B^{-1}\delta(y-y_0)\Pi_B + \delta(y-y_0)\frac{\partial B}{\partial y}\frac{\partial x}{\partial t}\right\}\end{aligned} \tag{17}$$

so

$$\frac{\partial B}{\partial t} = \varepsilon_B^{-1}\Pi_B\left(\frac{\partial x}{\partial y}\right)^{-1} + \frac{\partial B}{\partial y}\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial x}{\partial t} \tag{18}$$

which agrees with (16).

The evolution of Π_B is given by

$$\begin{aligned}\frac{\partial \Pi_B}{\partial t}(y_0,t) &= \mathrm{i}[\mathrm{H}_B, \Pi_B(y_0,t)]\\ &= -\int_0^L \mathrm{d}y \left(\frac{\partial x}{\partial y}\right)^{-1}\left\{\delta^{(1)}(y-y_0)\frac{\partial B}{\partial y} + \delta^{(1)}(y-y_0)\Pi_B\frac{\partial x}{\partial t}\right\}\end{aligned} \tag{19}$$

and so

$$\frac{\partial \Pi_B}{\partial t} = \frac{\partial}{\partial y}\left[\left(\frac{\partial x}{\partial y}\right)^{-1}\frac{\partial B}{\partial y} + \left(\frac{\partial x}{\partial y}\right)^{-1}\Pi_B\frac{\partial x}{\partial t}\right]$$
$$= \varepsilon\frac{\partial A}{\partial t}\frac{\partial^2 x}{\partial y \partial t} + \frac{\partial^2 A}{\partial x^2}\frac{\partial x}{\partial y} + \varepsilon\frac{\partial^2 A}{\partial x \partial t}\frac{\partial x}{\partial y}\frac{\partial x}{\partial t} + \frac{\partial \varepsilon}{\partial x}\frac{\partial A}{\partial t}\frac{\partial x}{\partial y}\frac{\partial x}{\partial t} \tag{20}$$

which from (12) should equal

$$\frac{\partial \Pi_B}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon_B\frac{\partial B}{\partial t}\frac{\partial x}{\partial y} - \varepsilon_B\frac{\partial B}{\partial y}\frac{\partial x}{\partial t}\right)$$
$$= \varepsilon\frac{\partial A}{\partial t}\frac{\partial^2 x}{\partial y \partial t} + \varepsilon\frac{\partial^2 A}{\partial x \partial t}\frac{\partial x}{\partial y}\frac{\partial x}{\partial t} + \varepsilon\frac{\partial^2 A}{\partial t^2}\frac{\partial x}{\partial y} + \frac{\partial \varepsilon}{\partial x}\frac{\partial A}{\partial t}\frac{\partial x}{\partial t}\frac{\partial x}{\partial y} + \frac{\partial \varepsilon}{\partial t}\frac{\partial A}{\partial t}\frac{\partial x}{\partial y}. \tag{21}$$

From (20) and (21) we can determine the evolution equation in terms of the original coordinate system:

$$\frac{\partial^2 A}{\partial x^2} = \varepsilon\frac{\partial^2 A}{\partial t^2} + \frac{\partial \varepsilon}{\partial t}\frac{\partial A}{\partial t} \tag{22}$$

which agrees with (3).

We can see that both the evolution of the system and the canonical commutation relation are independent of the coordinate system chosen.

4. Constructing p_n and q_n

In this section we directly construct the quantum operators p_n and q_n used in previous work and show that they have the properties attributed to them. These operators appear naturally in the abstract quantization used by Moore [1, 2] and are extremely useful in determining the quantum behaviour of the cavity given classical solutions.

For convenience, we will now denote differentiation by t with a dot and differentiation by x with a prime. As in the previous section, quantum operators will be denoted only by context.

Following Sarkar [2] we introduce the bilinear operator

$$\omega(A_1, A_2) = \int_0^{q(t)} \mathrm{d}x\, \varepsilon(x,t)\{\dot{A}_1(x,t)A_2(x,t) - A_1(x,t)\dot{A}_2(x,t)\} \tag{23}$$

where A_1 and A_2 are solutions to the wave equation in the cavity. ω is independent of time t because

$$\dot{\omega}(A_1, A_2) = \int_0^{q(t)} \mathrm{d}x\, \{A_1''(x)A_2(x) + \varepsilon\dot{A}_1(x)\dot{A}_2(x) - \varepsilon\dot{A}_1(x)\dot{A}_2(x) - A_1(x)A_2''(x)\}$$
$$= [A_1'(x)A_2(x) - A_1(x)A_2'(x)]_{x=0}^{x=q(t)}$$
$$= 0. \tag{24}$$

A_1 and A_2 can be either classical wavefunctions or the quantum operators.

We will choose classical solutions $u_n(x,t)$ and $v_n(x,t)$ such that

$$\omega(u_n, u_m) = \omega(v_n, v_m) = 0 \tag{25}$$
$$\omega(u_n, v_m) = \delta_{nm} \tag{26}$$

and any solution $A(x,t)$ can be written

$$A(x,t) = \sum_n (\alpha_n v_n(x,t) - \beta_n u_n(x,t)). \tag{27}$$

It is not particularly difficult to show that such solutions can always be constructed.

If we consider the solution with initial condition $A(x, t_0) = 0$, $\dot{A}(x, t_0) = -\varepsilon^{-1}(x, t_0)\delta(x - x_0)$ and choose α_n, β_n as in (27), then

$$\alpha_m = -\omega(A, u_m) = \int \mathrm{d}x\, \delta(x - x_0) u_m(x, t_0) = u_m(x_0, t_0) \tag{28}$$

$$\beta_m = -\omega(A, v_m) = \int \mathrm{d}x\, \delta(x - x_0) v_m(x, t_0) = v_m(x_0, t_0) \tag{29}$$

so

$$0 = \sum_n (u_n(x_0, t) v_n(x, t) - v_n(x_0, t) u_n(x, t)) \tag{30}$$

$$\varepsilon^{-1}(x, t)\delta(x - x_0) = \sum_n (v_n(x_0, t)\dot{u}_n(x, t) - u_n(x_0, t)\dot{v}_n(x, t)). \tag{31}$$

We now define operators

$$p_n = -\omega(A, u_n) \tag{32}$$

$$q_m = -\omega(A, v_m). \tag{33}$$

We relate p and q to A using (30) and (31):

$$\begin{aligned}
\sum_n (p_n v_n(x_0, t) - q_n u_n(x_0, t)) &= \sum_n \int_0^{q(t)} \mathrm{d}x\, \varepsilon(x, t)\{-\dot{A}(x, t) u_n(x, t) v_n(x_0, t) \\
&\quad + A(x, t)\dot{u}_n(x, t) v_n(x_0, t) + \dot{A}(x, t) v_n(x, t) u_n(x_0, t) \\
&\quad - A(x, t)\dot{v}_n(x, t) u_n(x_0, t)\} \\
&= A(x_0, t).
\end{aligned} \tag{34}$$

The canonical relations for p and q are then

$$\begin{aligned}
[p_n, q_m] &= [\omega(A, u_n), \omega(A, v_m)] \\
&= \int \mathrm{d}x\, \mathrm{d}x'\, \varepsilon(x)\varepsilon(x')\{[\dot{A}(x), \dot{A}(x')] u_n(x) v_m(x') - [\dot{A}(x), A(x')] u_n(x)\dot{v}_m(x') \\
&\quad - [A(x), \dot{A}(x')]\dot{u}_n(x) v_m(x') + [A(x), A(x')]\dot{u}_n(x)\dot{v}_m(x')\} \\
&= -\mathrm{i}\omega(u_n, v_m) \\
&= -\mathrm{i}\delta_{nm}.
\end{aligned} \tag{35}$$

5. Conclusion

We have shown that the ideal moving mirror cavity can be quantized in a conventional way when any appropriate transformation of Lagrangian coordinates is made, and that the quantum physics derived is independent of the specific choice of these coordinates. Any such quantization possesses a Hamiltonian which acts as the generator of time displacement in the usual way.

Also, we have shown that the canonical quantum operators introduced by Moore are consistent with this procedure. Consequently physical predictions given in [2] and [3] are unchanged.

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